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Optimal Control of the Coefficients of a Nonlinear Schrödinger-Type Equation

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Abstract—This paper is devoted to the solvability of an optimal control problem for the coefficient at the highest derivative and the quantum potential in a nonlinear and nonstationary Schrödinger-type equation. It generalizes the well-known equation of quantum mechanics. The simultaneous control problem for several coefficients of the state equation is considered, with a performance criterion specified by the residual of the boundary data of the solution. For this problem, well-posedness conditions are established and an existence theorem for its solution is proved. In addition, the optimization problem with a perturbed performance criterion is studied, and an existence and uniqueness theorem for its solution is proved. An explicit form of the first variation of the performance criterion is obtained, and an iterative algorithm for solving these problems is described. The results are novel for the standard Schrödinger equation as well.

Keywords: solvability of optimal control problem, Schrödinger-type equation, control of quantum processes, control of state equation coefficients

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1. INTRODUCTION

The Schrödinger-type equation generalizes the well-known Schrödinger equation of quantum mechanics and is used as a model in the theories of superconductivity and other fields of practice. This equation often arises in the diagnosis of nanostructured materials and atomic-molecular computing, when intra-atomic and intramolecular interaction potentials, laser pulses, and material characteristics need to be controlled. This equation is widely used in quantum information processing, adaptive optics and quasi-optics, bifurcation analysis of nonlinear models, investigation of magnetic quantum phenomena, and other applications; for example, see [1–9] and other publications. Linear and bilinear quantum systems, optimization problems of these systems with one control factor, with real or complex interaction potential, etc. were studied earlier. The finite difference method for numerically solving such control problems was developed in [9–11].

In modern practice it is often necessary to control many coefficients, the most influential quantum characteristics of nonlinear and nonstationary processes [1, 4–10]. Such control problems are most challenging for theoretical analysis and numerical solution: they are nonlinear, have an implicitly defined state operator, and often belong to the class of ill-posed problems [9, 10]; moreover, there are no direct methods for their observation and no formulation of performance criteria. The observation definition form is especially important for constructing performance criteria in quantum control processes since it must correspond to the quantum character of the processes under consideration. In this sense, boundary observation is convenient for measurements as well as for processing of results. It is crucial to investigate control processes for the coefficients at the higher derivatives of the state equation [1–3, 6–11]. Below, we study a simultaneous optimal control problem for the coefficient at the highest derivative and the quantum potential in a nonlinear nonstationary Schrödinger-type equation. The optimization problem with a “perturbed” performance criterion is also considered. Existence and uniqueness theorems for the solution of these problems are proved, and iterative algorithms for their solution are described. The main results are novel also for the standard Schrödinger equation of quantum mechanics.

2. PROBLEM STATEMENT

Let $l > 0$ and $T > 0$ be given numbers, $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, $\Omega_t = (0,l) \times (0,t)$, $\Omega = \Omega_T$, and $\psi(x,t)$ be a complex wave function with the spatial coordinate x and time t. The functional spaces used below were introduced, e.g., in [8, 11] and other publications. Accordingly, $L_p(0,l)$ is the Lebesgue space of measurable functions on $(0,l)$ that are integrable with degree $p \geqslant 1$, and $C^k[0,T;B)$ is the Banach space of $k \geqslant 0$ times continuously differentiable functions on $[0,T]$ whose values belong to a Banach space B.

Let $W_p^k(0,l)$ and $W_p^{k,m}(\Omega)$ be the Sobolev spaces of functions with generalized derivatives of orders $k \geqslant 0$ in x and $m \geqslant 0$ in t that are integrable with degree $p \geqslant 1$. We denote by $W_\infty^1(0,l) = \left\{w : w \in L_\infty(0,l), \frac{dw}{dx} \in L_\infty(0,l)\right\}$ the Banach space with the properties mentioned. The symbols $\forall^o$ and $\forall$ mean that the above properties hold for almost all and all, respectively, values of an appropriate variable. Positive constants independent of the values being estimated will be denoted by c_j, $j = 0,1,2,\dots$. From this point onwards, $a_i, b_i, s_i, i = 0,1,2,\dots$, are given positive numbers.

Consider a controlled process described by the following initial boundary-value problem for the Schrödinger-type equation:

$$i\rho^2 \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}\left(v_0(x)\frac{\partial \psi}{\partial x}\right) - v_1(x)\psi + a_1|\psi|^2\psi = f(x,t), \quad (x,t) \in \Omega, \tag{1}$$

$$\psi(x,0) = \varphi(x), \quad x \in (0,l), \tag{2}$$

$$\frac{\partial \psi(0,t)}{\partial x} = \frac{\partial \psi(l,t)}{\partial x} = 0, \quad t \in (0,T), \tag{3}$$

where $\rho > 0$ and a_1 are given real numbers; $\varphi(x)$ and $f(x,t)$ are given complex measurable functions satisfying the conditions $\varphi \in W_2^2(0,l)$, $\frac{d\varphi(0)}{dx} = \frac{d\varphi(l)}{dx} = 0$, and $f \in W_2^{0,1}(\Omega)$; the coefficients $v_0(x)$ and $v_1(x)$ of this equation are real control functions.

Assume that an additional boundary observation has the form

$$\psi(0,t) = y_0(t), \quad \psi(l,t) = y_1(t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

where $y_0 = y_0(t)$ and $y_1 = y_1(t)$ are given complex functions from the space $L_2(0,T)$. Let the vector function $v = v(x) = (v_0(x), v_1(x))$ be an element of the following set of admissible controls: $V \equiv \{v = (v_0, v_1) :\ v_0 \in W_2^1(0,l),\ v_1 \in L_2(0,l),\ 0 < b_0 \leqslant v_0(x) \leqslant b_1,\ \left|\frac{dv_0(x)}{dx}\right| \leqslant b_2,\ 0 < b_3 \leqslant v_1(x) \leqslant b_4,\ \forall^o x \in (0,l)\}$, where $b_j > 0$, $j = 0,1,\dots,4$, are given numbers. For each element $v \in V$, the wave function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ belonging to the space $B \equiv C^0([0,T]; W_2^1(0,l)) \cap C^1([0,T]; L_2(0,l))$ will be called the solution of the primal problem if it satisfies equation (1) for any $t \in [0,T]$ and almost all $x \in (0,l)$ and conditions (2) and (3) for almost all $x \in (0,l)$ and $t \in (0,T)$. The primal problem is an initial boundary-value problem for the Schrödinger-type equation (1). This problem was studied in [1–11] and other publications. As was established therein, under the above conditions, for each $v \in V$ it has a unique solution in the space B with the a priori upper bound

$$\|\psi(\cdot,t)\|_{W_2^1(0,l)} + \left\|\frac{\partial \psi(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \leqslant c_0\left(\|\varphi\|_{W_2^1(0,l)} + \|f\|_{L_2(0,T;W_2^1(0,l))}\right). \tag{5}$$

Now consider an optimal control problem for the function $v = v(x) = (v_0(x), v_1(x))$ in equation (1) with the initial boundary conditions (2) and (3): it is required to minimize the performance criterion

$$J_0(v) = \beta_0 \|\psi(0,\cdot) - y_0\|^2_{L^2(0,T)} + \beta_1 \|\psi(l,\cdot) - y_1\|^2_{L^2(0,T)} \tag{6}$$

on the set V, where the numbers $\beta_0 > 0$ and $\beta_1 > 0$ are given and $\beta_0 + \beta_1 = 1$. Note that problems with controls in state equation coefficients often have unstable solutions (see [9–12], etc.). Therefore, it is reasonable to study the problem with a perturbed performance criterion. Below we will minimize the criterion

$$J_\alpha(v) = \beta_0 \|\psi(0,\cdot) - y_0\|^2_{L^2(0,T)} + \beta_1 \|\psi(l,\cdot) - y_1\|^2_{L^2(0,T)} + \alpha \|v - \omega\|^2_H \tag{7}$$

on the set V under conditions (1)–(3), where $\alpha \geqslant 0$ is a given number, $H \equiv W_2^1(0,l) \times L_2(0,l)$ is the space of controls, and the element $\omega \in H$ is a given vector function. For the sake of brevity, problems (1)–(3), (6) and (1)–(3), (7) will be called problems (6) and (7), respectively.

3. SOLVABILITY OF THE OPTIMAL CONTROL PROBLEM

Theorem 1. *For any $\alpha \geqslant 0$, problem (7) has at least one solution.*

The proof is provided in the Appendix.

Let $H_\infty \equiv W_\infty^2(0,l) \times W_\infty^1(0,l)$ and $H_1 \equiv W_2^4(0,l) \times W_2^1(0,l)$. We study the optimal control problem on the set $V_1 \equiv \Big\{ v = (v_0, v_1); v_0 \in W_2^2(0,l), v_1 \in W_2^1(0,l), 0 < s_0 \leqslant v_0(x) \leqslant s_1, \left|\frac{dv_0(x)}{dx}\right| \leqslant s_2, \left|\frac{d^2 v_0(x)}{dx^2}\right| \leqslant s_3, 0 < s_4 \leqslant v_1(x) \leqslant s_5, \left|\frac{dv_1(x)}{dx}\right| \leqslant s_6, \forall x \in (0,l) \Big\}$, where $s_i > 0$, $i = 0,1,2,3,4,5,6$, are given numbers. Assume that the functions $\varphi(x)$ and $f(x,t)$ satisfy the conditions $\frac{d^3\varphi}{dx^3} \in L_2(0,l)$ and $\frac{\partial f}{\partial x} \in W_2^{0,1}(\Omega)$. Under these conditions, for each element $v = v(x)$ from the set V_1, the solution of the primal problem (1)–(3) exists, is unique for each $t \in [0,T]$, belongs to the space $B_1 = C^0([0,T]; W_2^4(0,l)) \cap C^1([0,T]; L_2(0,l))$, and obeys the a priori upper bound

$$\begin{aligned} &\|\psi(\cdot,t)\|_{W_2^1(0,l)} + \left\| \frac{\partial \psi(\cdot,t)}{\partial t} \right\|_{L_2(0,l)} + \left\| \frac{\partial^2 \psi(\cdot,t)}{\partial t^2} \right\|_{L_2(0,l)} \\ &\leqslant c_1 \left(\|\varphi\|_{W_2^1(0,l)} + \|f\|_{W_2^{1,1}(\Omega)} + \left\| \frac{\partial f}{\partial x} \right\|_{W_2^{1,1}(\Omega)} \right), \quad \forall t \in [0,T]. \end{aligned} \tag{8}$$

For details, see [9, 10].

Consider now an analog of problem (7) on the set $V_1 \subseteq H_1$: it is required to minimize the performance criterion

$$J_\alpha(v) = \beta_0 \|\psi(0,\cdot) - y_0\|^2_{L_2(0,T)} + \beta_1 \|\psi(l,\cdot) - y_1\|^2_{L_2(0,T)} + \alpha \|v - \omega\|^2_{H_1} \tag{9}$$

under conditions (1)–(3), where $\alpha \geqslant 0$ is a given number, $H_1 = W_2^2(0,l) \times W_2^1(0,l)$ is the space of controls, and $\omega \in H_1$ is a given vector function. It is easy to check that $V_1 \subseteq V$. If $\alpha = 0$, $V = V_1$, and $H = H_1$, problems (7) and (9) will coincide with problem (6). For $V = V_1$ and $H = H_1$, problems (7) and (9) become identical. Examples similar to those in [9, 10] demonstrate that, for $\alpha = 0$, the solution of problem (7) or (9) is unstable and non-unique. However, for $\alpha > 0$, we have the following result.

Theorem 2. *There exists an everywhere dense subset K in the space H_1 such that problem (9) has a unique solution for all $\alpha > 0$ and $\omega \in K$.*

The proof is provided in the Appendix.

4. CONCLUSIONS

The above theorems provide a basis for solving the problems under consideration. According to the existing experience in solving optimal control problems [2, 9, 10, 13, 16, 17], iterative numerical methods are the most effective tools for this purpose. Maple, Matlab, ANSYS, and similar software packages with visualization of the results are preferable and provide a convenient apparatus for numerical solution. Consider an iterative process for solving problem (7) or (9) based on the following numerical scheme of the conditional gradient method:

$$v^{(k+1)}(x) = v^{(k)}(x) + \lambda_k \left(w^{(k)}(x) - v^{(k)}(x)\right), \quad k = 0, 1, 2, \dots,$$

where $v^{(0)}(x)$ is an initial iteration step, which can be any element of the set V (or V_1); the algorithm parameter $\lambda_k \in (0,1)$ is chosen from the condition $J_0\left(v^{(k+1)}\right) < J_0\left(v^{(k)}\right)$; the element $w^{(k)}(x)$, $k = 1, 2, \dots$, is determined by minimizing the linear criterion

$$\delta J_0(v^{(k)}, w - v^{(k)}) + 2a\langle v^{(k)} + v^{(k-1)}, w - v^{(k)}\rangle \to \infty,$$

on the set V (or V_1), where $\delta J_0(v)$ is the first variation of the criterion $J_0(v)$.

Let $h \in H$ be an increment of control $v \in V$ such that $v + h \in V$. The first variation of the criterion $J_0(v)$ has the form

$$\delta J_0(v,h) = \int\limits_{\Omega} \left[\mathrm{Re}\left(\frac{\partial\psi(x,t)}{\partial x}\frac{\partial\varphi(x,t)}{\partial x}\right) h_0(x,t) + \mathrm{Re}\left(\psi(x,t)\varphi(x,t)\right) h_1(x,t)\right] dxdt,$$

where $\psi(x,t)$ and $\varphi(x,t)$ are the solutions of the primal and conjugate problems, respectively. Following the above scheme, we find a control sequence $(v^{(k)})$. For the problems under consideration, this sequence converges under sufficient conditions presented in [9, 10, 13, 14] and other publications. In addition, the interested reader can find therein wide classes of iterative processes based on different modifications of the gradient, Newton, and other methods for solving extremal problems as well as numerical analysis and regularization of the solution in the case of problem instability.

APPENDIX

Proof of Theorem 1. The existence of a minimizing sequence $(v'(x)) \in V$ for the solution of problem (7) follows from the boundedness from below of the criterion $J_\alpha(v)$. Let us denote $\psi_k = \psi_k(x,t) \equiv \psi(x,t;v^k)$, $k = 1, 2, \dots$, and $H_\infty \equiv W^1_\infty(0,l) \times L_\infty(0,l)$. Since the set V is a closed, bounded, and convex subset in the space H, it is possible to extract a subsequence from the sequence (v^k), denoted again by (v^k) for simplicity, so that $v_0^k \to v_0$ (*) weakly in $L_\infty(0,l)$, $\frac{dv_0^k}{dx} \to \frac{dv_0}{dx}$ (*) weakly in $L_\infty(0,l)$, and $v_1^k \to v_1$ (*) weakly in $L_\infty(0,I)$ as $k \to \infty$. Moreover, by the definition of the set V, it is (*) weakly closed in the space H. Due to the above limit relations and the space $W^k_\infty(0,l)$ embedded into $L_\infty(0,l)$, we obtain $v_0^k \to v_0$ strongly in $L_\infty(0,l)$ as $k \to \infty$.

The a priori bound (5) implies that the sequence $\{\psi_k(x,t)\}$ is uniformly bounded in the norm of B. Then it is possible to extract a subsequence from the sequence $\{\psi_k(x,t)\}$, denoted again by $\{\psi_k(x,t)\}$ for simplicity, so that $\psi_k(\cdot,t) \to \psi(\cdot,t)$ in $W^2_2(0,l)$ and $\frac{\partial\psi_k(\cdot,t)}{\partial t} \to \frac{\partial\psi(\cdot,t)}{\partial t}$ in $L_2(0,l)$ for $t \in [0,T]$ as $k \to \infty$. Since the space B is compactly embedded into $C^0([0,T]; W^2_0(0,l))$ (see [15]), we have $\|\psi_k(\cdot,l) - \pi(\cdot,l)\|_{W^2_0(0,l)} \to 0$ as $k \to \infty$ uniformly in $t \in [0,T]$. As is easily verified, each element of the sequence satisfies the identity

$$\begin{aligned}\int\limits_0^l &\left\{\frac{i\rho^2\partial\psi_k(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(v_o^k(x)\frac{\partial\psi_k(x,t)}{\partial x}\right) - v_1^k(x)\psi_k(x,t)\right\} \\ &+ a_1\|\psi_k(x,t)^2\|\psi(x,t) - f(x,t)\|\overline{\eta(x)}dx = 0, \quad k = 1, 2, \dots\end{aligned} \tag{A.1}$$

for all $t \in [0,T]$ and an arbitrary function $\eta \in L_2(0,l)$. Furthermore, $\psi_k(x,t) = 1, 2, \ldots$, satisfy the initial and boundary conditions (2) and (3). For each $t \in [0,T]$ and any function $\eta \in L_2(0,l)$, passing to the limit on $k = 1, 2, \ldots$ in identity (A.1) yields the same identity for the limit function $\psi_k(x,t)$. By analogy, it is straightforward to verify that the limit function $\psi_k(x,t)$ satisfies equation (1) for each $t \in [0,T]$ and almost all $x \in (0,l)$. For $t = 0$ we obtain $\|\psi_k(\cdot,0) - \psi(\cdot,0)\|_{L_2(0,l)} \to 0$ as $k \to \infty$. Therefore, as $k \to \infty$, passing to the limit in

$$\|\psi_k(\cdot,0) - \varphi\|_{L_2(0,l)} \leqslant \|\psi_k(\cdot,0) - \psi_k(\cdot,0)\|_{L_2(0,l)} + \|\psi_k(\cdot,0) - \varphi\|_{L_2(0,l)}$$

shows that the function $\psi(x,t)$ satisfies the initial condition (2) for almost all $x \in (0,l)$. Finally, we prove that the limit function $\psi_k(x,t)$ satisfies the boundary conditions (3). Indeed, for any functions from the space B, the limit relations $\frac{\partial \psi_k(s,\cdot)}{\partial x} \to \frac{\partial \psi(s,\cdot)}{\partial x}$, where $s = 0$ and $s = l$, are valid weakly in $L_2(0,T)$ as $k \to \infty$. With this relation, as $k \to \infty$, passing to the limit in the equality

$$\int_0^T \frac{\partial \psi(s,t)}{\partial x}\eta(t)dt = \int_0^T \left(\frac{\partial \psi(s,t)}{\partial x} - \frac{\partial \psi_k(s,t)}{\partial x}\right)\eta(t)dt + \int_0^T \frac{\partial \psi_k(s,t)}{\partial x}\eta(t)dt, \quad s = 0, l,$$

gives the expression

$$\int_0^T \frac{\partial \psi(s,t)}{\partial x}\eta(t)dt = 0, \qquad s = 0, l,$$

for any function $\eta(t)$ from $L_2(0,T)$. Hence, for almost all $t \in (0,T)$, the function $\psi(x,t)$ satisfies the boundary conditions (3). Thus, the limit function $\psi(x,t)$ of the sequence $(\psi_k(x,t))$ is a solution of the primal problem (1)–(3) for each control v from V. In addition, this function is the weak limit of the weakly convergent sequence $\{\psi_k(x,t)\}$ for each $t \in [0,T]$. Since the space B is compactly embedded into $C([0,l]; L_2(0,T))$, we have $\|\psi_k(x,t) - \psi(x,t)\|_{L_2(0,T)} \to 0$ as $k \to \infty$ for each $x \in [0,l]$. Consequently, for $x = 0$ and $x = l$, from the weak lower semicontinuity of the norm in $L_2(0,T)$ and the nonnegativity of the numbers β_0, β_l, and α, we arrive at the inequality $J_\alpha * \leqslant J_\alpha(v) \leqslant J_\alpha *$. In other words, the element $v = v(x) \in V$ is a solution of problem (7) for any $\alpha \geqslant 0$. The proof of Theorem 1 is complete.

Proof of Theorem 2. We check the continuity of the criterion $J_0(v)$ on the set V_1. Let $\delta\psi = \delta\psi(x,t) = \psi(x,t;\, v + \delta v) - \psi(x,t;\, v)$, where $\delta v \in H_1$ is an increment of an element $v \in V_1$ such that $v + \delta v \in V_1$, and let $\psi(x,t;\, v)$ be the solution of the primal problem (1)–(3) for element $v \in V_1$. Due to (1)–(3), the function $\delta\psi = \delta\psi(x,t)$ satisfies the equation

$$\begin{gathered} i\rho^2 \frac{\partial \delta\psi}{\partial t} + \frac{\partial}{\partial x}\Big((v_0(x) + \delta v_0(x))\frac{\partial \delta\psi}{\partial x}\Big) - (v_1(x) + \delta v_1(x))\delta\psi \\ + a_1\left(|\psi_\delta|^2 + |\psi|^2\right)\delta\psi + a_1\psi_\delta\psi\delta\overline{\psi} = -\frac{\partial}{\partial x}\Big(\delta v_0(x)\frac{\partial \psi}{\partial x}\Big) + \delta v_1(x)\psi, \quad (x,t) \in \Omega, \end{gathered} \tag{A.2}$$

with homogeneous initial boundary conditions. To estimate the solution of this initial boundary-value problem, we multiply both sides of equation (A.2) by the function $\delta\psi(x,t)$ and integrate the result over the domain Ω_t. In view of the homogeneity of the initial condition, subtracting from the resulting equality its complex conjugate yields

$$\begin{gathered} \|\rho\delta\psi(\cdot,t)\|^2_{L_2(0,l)} = 2a_1 \int \mathrm{Im}(\psi\psi_\delta, (\delta\overline{\psi})^2)\, dxd\tau \\ + 2\int \mathrm{Im}\left[\left(-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial \psi}{\partial x}\right) + \delta v_1(x)\psi\right)\delta\overline{\psi}\right] dxdt, \quad \forall t \in [0,T]. \end{gathered}$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality to this identity, we obtain

$$\|\rho\delta\psi(\cdot,t)\|^2_{L_2(0,l)} \leqslant 2|a_1|\int\limits_{\Omega_t} |\psi_\delta|\,|\psi|\,\delta\psi^2\,dx\,d\tau$$

$$+\int\limits_{\Omega_t}\left[-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right]^2 dx\,d\tau+\iint\limits_{\Omega_t}|\delta\psi|^2\ dx\,d\tau,\ \forall\,t\in[0,T]. \tag{A.3}$$

As is known [15], if a function $\varphi(.,t)\in W_2^1(0,l)$ is nonzero at the ends of the interval $[0,l]$, it satisfies the inequality

$$\|\varphi(\cdot,t)\|_{L_2(0,l)} \leqslant c\left\|\frac{\partial\varphi(\cdot,t)}{\partial x}\right\|^{\frac{1}{2}}_{L_2(0,l)}\|\varphi(.,t)\|^2_{L_2(0,l)}+d\,\|\varphi\|_{L_2(0,l)}\,,$$

where $c>0$, a, and $d>0$ are some constants. From this inequality and the a priori bound (8) for the solution of the primal problem (1)–(3), for each $V\in V_1$, it follows that

$$\|\psi(\cdot,t)\|^2_{L_2(0,l)} \leqslant c_4;\quad \|\psi^2(\cdot,t)\|^2_{L_2(0,l)} \leqslant c_4. \tag{A.4}$$

By these inequalities in equation (A.2), we have

$$\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)} \leqslant \left(2|a_1|c_4^2+1\right)\int\limits_0^t\|\delta\psi(\cdot,\tau)\|^2_{L_2(0,l)}\,d\tau$$

$$+\int\limits_{\Omega}\left|-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right|^2 dxd\tau,\quad \forall\,t\in[0,T].$$

Estimating the left-hand side of this inequalities gives

$$\int\limits_{\Omega_t}\left|-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right|^2 dxd\tau \leqslant c_5\int\limits_0^t\|\psi(\cdot,\tau)\|^2_{W_2^2(0,l)}\,d\tau\|\delta v\|^2_{B^1}.$$

Considering this inequality in equation (A.2), we obtain

$$\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)} \leqslant c_6\int\limits_0^t\|\delta\psi(\cdot,\tau)\|^2_{L_2(0,l)}\,d\tau+c_7\|\delta v\|^2_H,\quad \forall\,t\in[0,T].$$

With Gronwall's lemma applied to this inequality, it follows that

$$\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)} \leqslant c_8\|\delta v\|^2_{B_1},\quad \forall\,t\in[0,T]. \tag{A.5}$$

Now it is necessary to estimate the function $\frac{\partial\delta\psi}{\partial x}(x,t)$. For this purpose, we multiply both sides of equation (A.2) by $L(\delta\overline{\psi})=-\frac{\partial}{\partial x}\left((v_0(x)+\delta v_0(x))\frac{\partial\delta\overline{\psi}}{\partial x}\right)$ and integrate the resulting expression over the domain Ω_t :

$$\int\limits_{\Omega_t}\Bigg(i\rho^2\frac{\partial\delta\psi}{\partial t}L(\delta\psi)-|L(\delta\psi)|^2-(v_0(x)+\delta v_0(x))\delta\psi L(\delta\psi)$$

$$+a_1\left(|\psi_\delta|^2+|\psi|^2\right)\delta\psi L(\delta\overline{\psi})+a_1\psi_\delta\psi\delta\overline{\psi}L(\delta\overline{\psi})\Bigg)dxd\tau$$

$$=\int\limits_{\Omega}\left(\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)-\delta v_1(x)\psi\right)L(\delta\overline{\psi})dxd\tau,\quad \forall\,t\in[0,T].$$

We integrate on x in both sides of this equation and subtract from the resulting equality its complex conjugate:

$$\int\limits_{\partial_t} \rho^2 \frac{\partial}{\partial t}\Big(v_o(x) + \delta v_o(x)\Big)\Big|\frac{\partial \delta\psi}{\partial x}\Big|^2 \, dxd\tau$$

$$= 2\int\limits_{\Omega t} \left[\mathrm{Im}\Big(v_o(x) + \delta v_o(x)\Big)\left(\frac{dv_1(x)}{dx} + \frac{d\delta v_1(x)}{dx}\right)\delta\psi \frac{\partial \delta\overline{\psi}}{\partial x}\right] dxd\tau$$

$$- 2a_1 \int\limits_{\Omega_t} \left[\mathrm{Im}\Big(v_o(x) + \delta v_o(x)\Big)\frac{\partial}{\partial x}\Big(\big|\psi_\delta\big|^2 + \big|\psi\big|^2\Big)\delta\psi \frac{\partial \delta\overline{\psi}}{\partial x}\right] dxd\tau$$

$$- 2a_1 \int\limits_{\Omega_t} \left[\mathrm{Im}\Big(v_o(x) + \delta v_o(x)\Big)\frac{\partial}{\partial x}\Big(\psi_\delta \psi\Big)\delta\overline{\psi} \frac{\partial \delta\overline{\psi}}{\partial x}\right] dxd\tau$$

$$- 2a_1 \int\limits_{\Omega_t} \left[\mathrm{Im}\Big(v_o(x) + \delta v_o(x)\Big)\psi_\delta \psi \left(\frac{\partial \delta\overline{\psi}}{\partial x}\right)^2\right] dxd\tau$$

$$-2\int\limits_{\Omega_t} \left[\mathrm{Im}\Big(v_o(x) + \delta v_o(x)\Big)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\Big(\delta v_o(x)\frac{\partial \psi}{\partial x}\Big) - \delta v_1(x)\psi\right]\frac{\partial \delta\overline{\psi}}{\partial x}\right] dxd\tau, \quad \forall\, t \in [0, T].$$

Note that $v + \delta v \in V_1$ and $\delta\psi\,(x, 0) = 0,\ x \in (0, l)$. Due to the Cauchy–Bunyakovsky–Schwarz inequality and the bounds (A.4), we have

$$\begin{aligned}
s_0 \left\|\frac{\partial \delta\psi(\cdot, t)}{\partial x}\right\|^2_{L_2(0,l)} &\leqslant c_9 \int\limits_0^t \left\|\frac{\partial \delta\psi(\cdot, \tau)}{\partial x}\right\|^2_{L_2(0,l)} d\tau + c_{10} \int\limits_{\Omega_t} |\delta\psi|^2 \, dxd\tau \\
&+ 4s_1 \int\limits_{\Omega_t} |\delta v_0(x)|^2 \left|\frac{\partial^3 \psi}{\partial x^3}\right|^2 dxd\tau + 8s_1 \int\limits_{\Omega_t} \left|\frac{d\delta v_0(x)}{dx}\right|^2 \left|\frac{\partial^2 \psi}{\partial x^2}\right|^2 dxd\tau \\
&+ 4s_1 \int\limits_{\Omega_t} \left|\frac{d\delta v_1(x)}{dx}\right|^2 |\psi|^2 \, dxd\tau + 4s_1 \int\limits_{\Omega_t} \left(\left|\frac{d^2 \delta v_0(x)}{dx^2}\right| + |\delta v_1(x)|\right)^2 \left|\frac{\partial \psi}{\partial x}\right|^2 dxd\tau \\
&+ c_8\,|a_1|\,(2 + s_1) \int\limits_{\Omega_t} \left(\left|\frac{\partial \psi_\delta}{\partial x}\right| + \left|\frac{\partial \psi}{\partial x}\right|\right)^2 |\delta\psi|^2 \, dxd\tau,\ \forall t \in [0, T],
\end{aligned} \tag{A.6}$$

where $c_9 = s_1(|a_1|c_4 + |a_1|c_4^2 + 1 + s_5)$ and $c_{10} = s(|a_1|c_4^2 + s_5)$. Now it is necessary to estimate the last term on the right-hand side of this inequality. As is known [15], an arbitrary function $\varphi(\cdot, t) \in W_\lambda(0, l)$, $\forall\, t \in [0, T]$ satisfies the inequality

$$\left\|\frac{\partial \varphi(\cdot, t)}{\partial x}\right\|_{L_\infty(0,l)} \leqslant c_0 \left\|\frac{\partial \varphi(\cdot, t)}{\partial x}\right\|_{L_2(0,l)} \|\varphi(\cdot, t)\|_{L_2(0,l)}, \quad \forall\, t \in [0, T], \tag{A.7}$$

where $c_0 > 0$ is some constant. With the functions $\frac{\partial \psi_\delta(x,t)}{\partial x}$ and $\frac{\partial \psi(x,t)}{\partial x}$ taken instead of the function $\varphi(x, t)$ in this inequality, we utilize (A.5) and (A.6) to obtain

$$\frac{\partial \psi_\delta(\cdot, t)}{\partial x}\Big|_{L_\infty(0,l)} \leqslant c_{11}, \qquad \left|\frac{\partial \psi(x, t)}{\partial x}\right|_{L_\infty(0,l)} \leqslant c_{11}. \tag{A.8}$$

Using these bounds in (7) and inequality (A.5), we have

$$\left\|\frac{\partial\delta\psi(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)} \leqslant c_{12}\int\limits_0^t \left\|\frac{\partial\psi(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)} d\tau + c_{13}\left\|\delta v\right\|^2_{B_1}.$$

With Gronwall's lemma applied to this inequality, it follows that

$$\left\|\frac{\partial\psi(x,t)}{\partial x}\right\|^2_{L_2(0,l)} \leqslant c_{15}\|\delta v\|^2_{B_1}, \quad \forall\, t\in[0,T].$$

Combining this bound with (A.5) gives

$$\left\|\delta\psi(\cdot,t)\right\|^2_{W_2^1(0,l)} \leqslant c_{15}\|\delta v\|^2_{B_1}, \quad \forall\, t\in[0,T]. \tag{A.9}$$

Let us transform the increment of the criterion $J_0(v)$ on an arbitrary element $v\in V_1$:

$$\Delta J_0(\nu) = J_0(\nu+\delta\nu) - J_0(\nu) = 2\beta_0\int\limits_0^T \mathcal{R}\epsilon\left[(\psi(0,t)-y_0(t))\,\delta\psi(0,t)\right]\,dt$$
$$+2\beta_1\int\limits_0^t \mathcal{R}\epsilon\left[(\psi(l,t)-y_1(t))\,\delta\psi(l,t)\right]\,dt + \beta_0\left\|\delta\psi(x,t)\right\|^2_{L_2(0,T)} + \beta_1\left\|\delta\psi(x,t)\right\|^2_{L_2(0,T)}. \tag{A.10}$$

By the embedding theorem, for the trace of functions from the space $W_2^{1,0}(\Omega)$, we have

$$\|\psi(0,\cdot)\|^2_{L_2(0,T)} + \|\psi(l,\cdot)\|^2_{L_2(0,T)} \leqslant c_{16}\|\psi\|^2_{W_2^{1,0}(\Omega)},$$
$$\|\delta\psi(0,\cdot)\|^2_{L_2(0,T)} + \|\delta\psi(l,\cdot)\|^2_{L_2(0,T)} \leqslant c_{16}\|\delta\psi\|^2_{W_2^{1,0}(\Omega)}.$$

These inequalities imply

$$\|\psi(0,\cdot)\|^2_{L_2(0,T)} + \|\psi(l,\cdot)\|^2_{L_2(0,T)} \leqslant c_{17}, \tag{A.11}$$
$$\|\delta\psi(0,\cdot)\|^2_{L_2(0,T)} + \|\delta\psi(l,\cdot)\|^2_{L_2(0,T)} \leqslant c_{17}. \tag{A.12}$$

Let us apply the Cauchy–Bunyakovsky–Schwarz inequality to the expression (A.10) and use (A.11) and (A.12). In view of the condition $y_0, y_1\in L_2(0,l)$, we finally derive the upper bound

$$|\Delta J_0(v)| \leqslant c_{19}(\|\delta v\|_{B^1} + \|\delta v\|^2_{B_1}), \quad v\in V_1.$$

According to this inequality, the criterion $J_0(v)$ is continuous on the set V_1. The criterion $J_0(v)$ is positive: $J_0(v)\geqslant 0$ for $v\in V_1$. Furthermore, the set V_1 is a closed, bounded, and convex set in the Hilbert space H_1. Hilbert spaces are uniformly convex. If a functional $I_0(v)$ is lower semicontinuous and bounded from below on a closed bounded set $U\subset X$ of a uniformly convex Banach space X, then there exists an everywhere dense subset $K\subset X$ such that, for any $w\in K$ and any $\alpha>0$, the functional $I_0(v)+\alpha\|v-w\|^2_X$ achieves the minimum value on a single element of the set $K\cap U$ [16]. For problem (9), all conditions and statements of this theorem are valid, and its proof is complete.

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